

# A SINGULAR DEMAILLY-PĂUN THEOREM

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**ABSTRACT.** We give a numerical characterization of the Kähler cone of a possibly singular compact analytic variety which is embedded in a smooth ambient space.

## 1. INTRODUCTION

The classical Nakai-Moishezon ampleness criterion (see e.g. [8] and references therein) characterizes ample line bundles on a projective variety as those which have positive intersection against all subvarieties. This was later extended to  $\mathbb{R}$ -divisors by Campana-Peternell [1]. In a groundbreaking paper, Demailly and Păun [7] proved a vast generalization of this result, which holds for all real  $(1, 1)$  classes on a compact Kähler manifold. More precisely, they proved that the Kähler cone of a compact Kähler manifold is one of the connected components of the positive cone, consisting of classes which have positive intersection against all analytic subvarieties. Very recently, a new proof of this theorem was obtained by combining the main result of our previous work [3] with a result of Chiose [2].

In this note, we prove an extension of the Demailly-Păun theorem [7] to singular varieties which are embedded in a smooth ambient space. A  $(1, 1)$  class on the variety is just taken to be the restriction of a  $(1, 1)$  class from the ambient space, and such a class is Kähler if it is so in a neighborhood of the variety inside the ambient space. This is in fact equivalent to the more intrinsic definition of a Kähler class on a compact analytic space as given for example in [13], as shown by Păun [9], and this allows us to avoid discussing these more technical notions. With these observations in mind, our main theorem is the following:

**Theorem 1.1.** *Let  $(M, \omega)$  be a smooth (but possibly noncompact and incomplete) Kähler manifold, and  $E \subset M$  be a compact analytic subvariety. Let  $\alpha$  be a closed smooth real  $(1, 1)$  form on  $M$  such that*

$$(1.1) \quad \int_V \alpha^k \wedge \omega^{\dim V - k} > 0,$$

*for all positive-dimensional irreducible analytic subvarieties  $V \subset E$ , and for all  $1 \leq k \leq \dim V$ . Then there exist an open neighborhood  $U$  of  $E$  in  $M$  and a smooth function  $\varphi : U \rightarrow \mathbb{R}$  such that  $\alpha + \sqrt{-1}\partial\bar{\partial}\varphi$  is a Kähler metric on  $U$ . If  $M$  is an open subset of the regular locus of some projective variety,*

then the inequalities

$$(1.2) \quad \int_V \alpha^{\dim V} > 0,$$

for all  $V$  as above suffice to reach the same conclusion.

This theorem answers a question that was posed to us by R.J. Conlon and H.-J. Hein, in relation to their paper [5] (see also [4, 1.3.5]). Applications of this result to the study of the Kähler cone of asymptotically conical Calabi-Yau manifolds will appear in a forthcoming revision of [5].

The main tools we use are the Demailly-Păun theorem itself, for smooth compact Kähler manifolds, and our recent theorem [3] which shows that the non-Kähler locus of a nef and big class on a compact complex manifold equals the null locus of the class. The idea is to work by induction on the dimension on  $E$  (as in [7]), and to prove the result by working on a resolution of singularities (as in [3]). This way we avoid any technical discussion of currents on singular analytic spaces.

In future work, we hope to address the extension of the Demailly-Păun theorem [7] as well as the main result of our previous work [3] to general compact Kähler (reduced and irreducible) analytic spaces.

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## 2. PROOF OF THEOREM 1.1

This section contains the proof of Theorem 1.1.

Clearly we may assume that no component of  $E$  is zero-dimensional, since for those the result is trivial.

Let us first assume that  $E$  is irreducible and 1-dimensional. Let  $\nu : \tilde{M} \rightarrow M$  be an embedded resolution of singularities of  $E \subset M$ , so that  $\tilde{M}$  is smooth, connected and Kähler, and the proper transform  $\tilde{E}$  of  $E$  is a smooth compact Riemann surface. We will also write  $\nu : \tilde{E} \rightarrow E$  for the induced map, so that  $\nu^*\alpha$  is a smooth closed real  $(1, 1)$  form with  $\int_{\tilde{E}} \nu^*\alpha > 0$ . Therefore the class  $[\nu^*\alpha]$  on  $\tilde{E}$  is Kähler, and we can find a smooth function  $\psi$  on  $\tilde{E}$  such that  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\psi > 0$  on  $\tilde{E}$ . It is elementary to find an open neighborhood  $\tilde{U}$  of  $\tilde{E}$  in  $\tilde{M}$  and a smooth extension of  $\psi$  to  $\tilde{U}$  (still denoted by  $\psi$ ) such that  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\psi > 0$  on  $\tilde{U}$  (see e.g. [9, Lemme 1, p.416]). Note that  $U = \nu(\tilde{U}) \setminus E_{sing}$  is an open neighborhood of  $E_{reg}$  inside  $M$ , but in general it is not the case that  $\nu(\tilde{U})$  is an open neighborhood of  $E$ .

inside  $M$ , because it may “pinch off” near  $E_{sing}$ . Furthermore, even if  $\nu(\tilde{U})$  happens to be an open neighborhood of  $E$ , the pushforward function  $\nu_*\psi$  is not well-defined wherever different branches of  $\tilde{E}$  come together under the map  $\nu$ . Therefore, we need to work a bit harder to achieve our goal.

Let  $\{p_1, \dots, p_N\} \subset \tilde{E}$  be the exceptional locus of  $\nu$  intersected with  $\tilde{E}$ , so that  $\{\nu(p_1), \dots, \nu(p_N)\}$  equals the singular set of  $E$ . For each point  $p_j$  we add to  $\psi$  a function of the form  $\varepsilon\theta(z) \log|z - p_j|$ , where  $\varepsilon > 0$  is small enough, where  $z = (z_1, \dots, z_N)$  are local coordinates for  $\tilde{M}$  near  $p_j$ , and  $\theta$  is a smooth cutoff function supported in a small neighborhood of  $p_j$  in  $\tilde{M}$ , so that we obtain a new function  $\tilde{\psi}$ , which is smooth away from the  $p_j$ 's and goes to  $-\infty$  there, and such that  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\tilde{\psi}$  is a Kähler current on  $\tilde{U}$ .

Then the smooth function  $\hat{\psi} = \nu_*\tilde{\psi}$  on  $U$  satisfies  $\alpha + \sqrt{-1}\partial\bar{\partial}\hat{\psi} > 0$ , but we are not done yet because  $U$  does not contain the singular points of  $E$ . Let  $\{\nu(p_1), \dots, \nu(p_k)\}$  be all the singular points of  $E$  (so  $k \leq N$ ), and fix charts  $U_j$  for  $M$  centered at  $\nu(p_j)$  for  $1 \leq j \leq k$ , with coordinates so that each  $U_j$  is the Euclidean ball of radius 2. Call  $U'_j$  the Euclidean ball of radius 1 in these coordinates, and let  $A$  be the minimum of  $\hat{\psi}$  on the compact set

$$\bigcup_{j=1}^k \overline{(\partial U'_j) \cap U},$$

which is a finite number because  $\hat{\psi}$  is smooth there. Choose a large constant  $B > 0$  such that on each  $U_j$  we have  $\alpha + B\sqrt{-1}\partial\bar{\partial}|z|^2 > 0$ . On  $U \cap U_j$  then we have that  $\hat{\psi}$  and  $B|z|^2 + A - B - 1$  are both strictly  $\alpha$ -plurisubharmonic, with  $\hat{\psi}$  approaching  $-\infty$  at the center of the ball  $U_j$ , and with  $\hat{\psi} > B|z|^2 + A - B - 1$  on a neighborhood of  $(\partial U'_j) \cap U$ . If  $\widetilde{\max}$  denotes a regularized maximum function (see, e.g. [6, I.5.18]), then

$$\psi_g = \widetilde{\max}(\hat{\psi}, B|z|^2 + A - B - 1)$$

is smooth and strictly  $\alpha$ -plurisubharmonic on  $U_j \cap U$ , it equals  $\hat{\psi}$  in a neighborhood of  $(\partial U'_j) \cap U$ , and it equals  $B|z|^2 + A - B - 1$  as we approach the origin. Therefore the function  $\psi_g$  trivially glues to  $\hat{\psi}$  outside  $U'_j$ , and we can extend it to be equal to  $B|z|^2 + A - B - 1$  in a small neighborhood of the origin in  $U_j$ . Repeating this construction for all  $j$ , and gluing each of them to  $\hat{\psi}$ , we finally obtain an open neighborhood  $\overline{U}$  of  $E$  in  $M$  and a smooth function  $\varphi$  on  $\overline{U}$  such that  $\alpha + \sqrt{-1}\partial\bar{\partial}\varphi$  is a Kähler metric on  $\overline{U}$ , as required.

Next, we assume that  $E$  has pure dimension 1, but need not be irreducible anymore. Then, writing  $E = \cup_j E_j$  with  $E_j$  irreducible, we can apply the result to each  $E_j$  and obtain  $U_j, \varphi_j$  as above, and “glue” them all together using [9, Lemme, p.419], and obtain the desired Kähler potential  $\varphi$  on some neighborhood  $\overline{U}$  of  $E$ .

We now deal with the general case, by induction on  $\dim E$  (which is by definition the max of the dimensions of the irreducible components of  $E$ ). The base of the induction is what we have just proved. For the induction

step, let  $\dim E = n$  and assume the result holds in all dimensions  $< n$ . As we just did, it is enough to prove the theorem in the case when  $E$  is irreducible, since if there are several components then we work on each one separately, and in the end glue the resulting metrics as before. So we will assume that  $E$  is irreducible. Take  $\nu : \tilde{M} \rightarrow M$  to be an embedded resolution of singularities of  $E \subset M$ , obtained as a composition of blowups with smooth centers, so that  $\tilde{M}$  is smooth and Kähler, and the proper transform  $\tilde{E}$  of  $E$  is smooth.

Then  $\nu^*\alpha$  is a smooth closed real  $(1,1)$  form on  $\tilde{E}$ , and we claim that its class  $[\nu^*\alpha]$  on  $\tilde{E}$  is nef. If assume that  $M$  is an open subset of the regular locus of some projective variety, then this holds because we have  $\int_V (\nu^*\alpha)^{\dim V} \geq 0$  for all positive-dimensional irreducible subvarieties  $V$  in  $\tilde{E}$  (using (1.1)), and so [7, Theorem 4.5(ii)] gives that the class  $[\nu^*\alpha]$  on  $\tilde{E}$  is nef. However, in our general setup (where there may be no projective compactification), to use [7, Theorem 4.3(ii)] we would have to check instead that

$$\int_V \nu^*\alpha^k \wedge \tilde{\omega}^{\dim V - k} \geq 0,$$

for all positive-dimensional irreducible subvarieties  $V \subset \tilde{E}$ , for some Kähler form  $\tilde{\omega}$  on  $\tilde{E}$  and for all  $1 \leq k \leq \dim V$ , and it does not seem easy to check this directly. Instead, we argue as follows. We have

$$\int_{\tilde{E}} \nu^*(\alpha^k \wedge \omega^{\dim E - k}) > 0,$$

for  $1 \leq k \leq \dim E$ , because  $\nu : \tilde{E} \rightarrow E$  is a modification, and using (1.1). Since the class  $[\nu^*\omega]$  is nef on  $\tilde{E}$ , we can find Kähler classes on  $\tilde{E}$  arbitrarily close to it, and therefore there exists a Kähler metric  $\tilde{\omega}$  on  $\tilde{E}$  such that

$$(2.1) \quad \int_{\tilde{E}} \nu^*\alpha^k \wedge \tilde{\omega}^{\dim E - k} > 0,$$

for  $1 \leq k \leq \dim E$ . Now for  $t \geq 0$  sufficiently large, the class  $[\nu^*\alpha + t\tilde{\omega}]$  is Kähler on  $\tilde{E}$ . Let  $t_0$  be the minimum value of  $t$  such that the class  $[\nu^*\alpha + t\tilde{\omega}]$  is nef on  $\tilde{E}$ , and suppose for a contradiction that  $t_0 > 0$ . By definition the class  $[\nu^*\alpha + t_0\tilde{\omega}]$  is not Kähler on  $\tilde{E}$ . Thanks to [7, Theorem 0.1], there exists a positive-dimensional irreducible analytic subvariety  $V \subset \tilde{E}$ , such that

$$(2.2) \quad \int_V (\nu^*\alpha + t_0\tilde{\omega})^{\dim V} = 0,$$

since if we had strict positivity for all such  $V$  then the class  $[\nu^*\alpha + t_0\tilde{\omega}]$  would be Kähler. Also  $V$  must be properly contained in  $\tilde{E}$ , because we have

$$\int_{\tilde{E}} (\nu^*\alpha + t_0\tilde{\omega})^{\dim E} > 0,$$

by (2.1). Then  $\nu(V)$  is an irreducible analytic subvariety of  $E$  (possibly a point), of dimension strictly less than  $\dim E$ , and with the same positivity property (1.1), so by induction we can find an open neighborhood  $W$  of

$\nu(V)$  in  $M$  and a smooth function  $\eta$  on  $W$  such that  $\alpha + \sqrt{-1}\partial\bar{\partial}\eta > 0$ . Therefore, in the open neighborhood  $\nu^{-1}(W)$  of  $V$  the smooth function  $\nu^*\eta$  satisfies  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}(\nu^*\eta) \geq 0$ . Since  $\tilde{\omega}$  is Kähler on  $\tilde{E}$  and  $t_0 > 0$ , this implies that

$$\int_V (\nu^*\alpha + t_0\tilde{\omega})^{\dim V} > 0,$$

contradicting (2.2). Therefore we must have  $t_0 \leq 0$ , and so the class  $[\nu^*\alpha]$  is indeed nef on  $\tilde{E}$ .

This proves our claim that the class  $[\nu^*\alpha]$  is nef on  $\tilde{E}$ , and since

$$\int_{\tilde{E}} (\nu^*\alpha)^{\dim E} = \int_E \alpha^{\dim E} > 0,$$

by (1.1), we can apply [7, Theorem 2.12] and see that this class is also big, i.e. it contains a Kähler current  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\psi$ , which we may assume has analytic singularities thanks to Demailly's regularization theorem (see [7, Theorem 3.2]). Also, if  $V \not\subset \text{Exc}(\nu) \cap \tilde{E}$  then  $\nu(V)$  is an irreducible subvariety of  $E$  of the same dimension as  $V$ , and  $\nu : V \rightarrow \nu(V)$  is bimeromorphic and so we have  $\int_V (\nu^*\alpha)^{\dim V} = \int_{\nu(V)} \alpha^{\dim V} > 0$ , thanks to assumption (1.1). This means that the null locus of the class  $[\nu^*\alpha]$  on  $\tilde{E}$  is contained in  $\text{Exc}(\nu)$ , and so using [3, Theorem 1.1], we may choose  $\psi$  to be smooth on  $\tilde{E} \setminus \text{Exc}(\nu)$ . We use [7, Lemma 2.1] to obtain a quasi-plurisubharmonic function with nontrivial analytic singularities along  $\text{Exc}(\nu)$ , and add a small multiple of it to  $\psi$ , to obtain a function  $\tilde{\psi}$  which is smooth on  $\tilde{E} \setminus \text{Exc}(\nu)$  and goes to  $-\infty$  along  $\text{Exc}(\nu)$ , and such that  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\tilde{\psi}$  is a Kähler current on  $\tilde{E}$  with analytic singularities along  $\text{Exc}(\nu)$ .

As in the first part of the proof of [3, Theorem 3.2], up to modifying  $\tilde{\psi}$  slightly (maintaining its same properties) we can find an extension  $\tilde{\psi}'$  to an open neighborhood  $\tilde{U}$  of  $\tilde{E} \setminus \text{Exc}(\nu)$  in  $\tilde{M}$ .

Here are some details for this construction (see [3] for full details). By a resolution of singularities argument, we construct a modification  $\mu : \hat{M} \rightarrow \tilde{M}$ , which is a composition of blowups with smooth centers, such that  $\mu(\text{Exc}(\mu))$  is equal to  $\text{Exc}(\nu)$ , such that the proper transform  $\hat{E}$  of  $\tilde{E}$  is smooth, and the pullback under  $\mu$  of the ideal sheaf on  $\tilde{E}$  which defines the singularities of the Kähler current  $\nu^*\alpha + \sqrt{-1}\partial\bar{\partial}\tilde{\psi}$  is a principal ideal, supported along a simple normal crossings divisor, which is the restriction to  $\hat{E}$  of a simple normal crossings divisor on  $\hat{M}$  (which is equal to  $\text{Exc}(\mu)$ ), which has normal crossings with  $\hat{E}$ . We then cover  $\hat{E}$  by finitely many coordinate charts  $\{W_j\}$  for  $\hat{M}$ . To the pullback  $\mu^*\tilde{\psi}'$  we add a small multiple of  $\sqrt{-1}\partial\bar{\partial} \log |s|_h^2$ , where  $s$  defines  $\text{Exc}(\mu)$  (and  $h$  is chosen suitably), to obtain a strictly  $\mu^*\nu^*\alpha$ -plurisubharmonic function  $\Psi$  on  $\hat{E}$ , with analytic singularities as before (in particular, smooth away from  $\text{Exc}(\mu)$ ). For each  $j$ , we then extend  $\Psi|_{W_j \cap \hat{E}}$  to a function  $\psi_j$  on  $W_j$  in an elementary fashion, still

preserving strict  $\mu^*\nu^*\alpha$ -plurisubharmonicity. Then we use a gluing procedure inspired by a classical method of Richberg [11] (see e.g. [12, Lemma 3.3]), but with the extra difficulty that now the functions  $\psi_j$  have poles. Nevertheless, arguing exactly as in [3, Proof of Theorem 3.2], we can obtain an open neighborhood  $U_1$  of  $\hat{E}$  in  $\tilde{M}$  and a strictly  $\mu^*\nu^*\alpha$ -plurisubharmonic function  $\tilde{\Psi}$  on  $U_1$ , which restricts to  $\Psi$  on  $\hat{E}$ , and is smooth on  $\hat{E} \setminus \text{Exc}(\mu)$ .

Here we highlight that since  $\hat{E}$  is a complex submanifold of a complex manifold, constructing this extension  $\tilde{\Psi}$  on an open neighborhood  $U_1$  of  $\hat{E}$  would be standard by Richberg [11] if  $\Psi$  was smooth (or even just continuous) on  $\hat{E}$ . On the other hand, if the singularities of  $\Psi$  were completely arbitrary, then such an extension would not be possible in general. The key property that saves us here is that the singular locus of  $\Psi$  is the intersection with  $\hat{E}$  of a simple normal crossings divisor,  $\text{Exc}(\mu)$ , in the ambient space  $\tilde{M}$ .

Then we take  $\tilde{U} = \mu(U_1)$ , and  $\tilde{\psi}' = \mu_*\tilde{\Psi}$ , which are as required. In particular,  $\tilde{U}$  is an open neighborhood of  $\hat{E} \setminus \text{Exc}(\nu)$  in  $\tilde{M}$ , and  $\tilde{\psi}'$  is strictly  $\nu^*\alpha$ -plurisubharmonic, and it is smooth on  $\tilde{U} \setminus \text{Exc}(\nu)$ .

On the open set  $U = \nu(\tilde{U}) \setminus E_{\text{sing}}$  (which is a neighborhood of  $E_{\text{reg}}$  in  $M$ ) we have the smooth function  $\hat{\psi} = \nu_*\tilde{\psi}'$  with  $\alpha + \sqrt{-1}\partial\bar{\partial}(\nu_*\tilde{\psi}')$  a smooth Kähler metric there, and with  $\nu_*\tilde{\psi}'$  approaching  $-\infty$  along  $E_{\text{sing}}$ . Now  $E_{\text{sing}}$  is a subvariety of  $M$  of dimension strictly less than  $n$ , with the same positivity property (1.1), so by induction we can find an open neighborhood  $W$  of  $E_{\text{sing}}$  in  $M$  and a smooth function  $\hat{\varphi}$  on  $W$  with  $\alpha + \sqrt{-1}\partial\bar{\partial}\hat{\varphi} > 0$  on  $W$ . We may also assume that  $\hat{\varphi}$  is defined on a slightly larger open set, so that it is smooth up to  $\partial W$ .

If we let  $A$  be the minimum of  $\hat{\psi}$  on the compact set  $\overline{(\partial W) \cap U}$  and  $B$  be the maximum of  $\hat{\varphi}$  on the same set, then  $\hat{\psi} > \hat{\varphi} + A - B - 1$  holds on a neighborhood of  $(\partial W) \cap U$ . Then

$$\psi_g = \widetilde{\max}(\hat{\psi}, \hat{\varphi} + A - B - 1)$$

is smooth and strictly  $\alpha$ -plurisubharmonic on  $U \cap W$ , equal to  $\hat{\psi}$  near  $(\partial W) \cap U$ , and equal to  $\hat{\varphi} + A - B - 1$  as we approach  $E_{\text{sing}}$ . Therefore  $\psi_g$  trivially glues to  $\hat{\psi}$  outside  $W$ , and we can extend it to be equal to  $\hat{\varphi} + A - B - 1$  in a neighborhood of  $E_{\text{sing}}$ . In this way we obtain an open neighborhood  $\overline{U}$  of  $E$  in  $M$  and a smooth function  $\varphi$  on  $\overline{U}$  such that  $\alpha + \sqrt{-1}\partial\bar{\partial}\varphi$  is a Kähler metric on  $\overline{U}$ , as required.

Lastly, the statement in the projective case follows from the Kähler one exactly as in [7], by choosing  $\omega$  to be the curvature form of a very ample line bundle  $L$  on the projective variety which contains  $M$  as an open subset, and observing that

$$\int_V \alpha^k \wedge \omega^{\dim V - k} = \int_{V \cap H_1 \cap \dots \cap H_{\dim V - k}} \alpha^k,$$

for generic members  $H_1, \dots, H_{\dim V-k}$  of the linear system  $|L|$ , so that  $V \cap H_1 \cap \dots \cap H_{\dim V-k}$  is an irreducible subvariety of dimension  $k$ .

## REFERENCES

- [1] F. Campana, T. Peternell *Algebraicity of the ample cone of projective varieties*, J. Reine Angew. Math. **407** (1990), 160–166.
- [2] I. Chiose *The Kähler rank of compact complex manifolds*, preprint, arXiv:1308.2043.
- [3] T. Collins, V. Tosatti *Kähler currents and null loci*, to appear in Invent. Math.
- [4] R.J. Conlon, H.-J. Hein *Asymptotically conical Calabi-Yau manifolds, I*, Duke Math. J. **162** (2013), no. 15, 2855–2902.
- [5] R.J. Conlon, H.-J. Hein *Asymptotically conical Calabi-Yau manifolds, III*, preprint, arXiv:1405.7140.
- [6] J.-P. Demailly *Complex Analytic and Differential Geometry*, available on the author’s webpage.
- [7] J.-P. Demailly, M. Păun *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math., **159** (2004), no. 3, 1247–1274.
- [8] S.L. Kleiman *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344.
- [9] M. Păun *Sur l’effectivité numérique des images inverses de fibrés en droites*, Math. Ann. **310** (1998), no. 3, 411–421.
- [10] D.H. Phong, J. Sturm *On the singularities of the pluricomplex Green’s function*, in *Advances in analysis. The legacy of Elias M. Stein*, 419–435, Princeton University Press, 2014.
- [11] R. Richberg *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1968), 257–286.
- [12] P.A.N. Smith *Smoothing plurisubharmonic functions on complex spaces*, Math. Ann. **273** (1986), no. 3, 397–413.
- [13] J. Varouchas *Kähler spaces and proper open morphisms*, Math. Ann. **283** (1989), no. 1, 13–52.

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